

MANIFOLDS IN WHICH THE POINCARÉ CONJECTURE IS TRUE

BY

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1. **Introduction.** A *homotopy 3-cell* is a compact, connected triangulated 3-manifold whose boundary is a 2-sphere and whose fundamental group is trivial. The Poincaré conjecture is true in dimension three if and only if every homotopy 3-cell is a 3-cell. If M is a 3-manifold, one says that the *Poincaré conjecture is true in M* provided that every imbedded homotopy 3-cell in M is a 3-cell. Equivalently, one says that M is in PC (the “Poincaré Category”).

It will be shown in this paper that whenever any 3-manifold in PC is pasted to itself or to any other 3-manifold in PC across a disk, a sphere, or a projective plane, the resulting 3-manifold is in PC. Moreover, it will be shown that if pasting across any compact 2-manifold other than a disk, a sphere, or a projective plane always preserves membership in PC, then the Poincaré conjecture is true.

It follows from Theorem 1 of Moise [6] that if the underlying space of a homotopy 3-cell can be topologically imbedded in a 3-manifold M , then there is a piecewise linear imbedding of that homotopy 3-cell into any triangulation of M (and, of course, M can be triangulated). Therefore, if every piecewise linearly imbedded homotopy 3-cell in any particular triangulation of M is a 3-cell, then M is in PC.

From now on, any manifold considered will be triangulated, and any map considered will be piecewise linear.

2. **Examples of 3-manifolds in PC.** Explicit consideration of PC probably begins with Brown [1], in which these four examples are given.

(2.1) Euclidean 3-space R^3 is in PC.

(2.2) If B is a 3-manifold in PC, then every 3-dimensional submanifold of B is in PC.

(2.3) If a 3-manifold B has a covering space which is in PC, then B is in PC.

(2.4) If M is a 2-manifold, then the 3-manifold $M \times [0, 1]$ is in PC.

The following examples further indicate the extent of PC.

(2.5) The 3-sphere S^3 is in PC.

Proof. This follows from Alexander's Theorem.

(2.6) The lens spaces are in PC.

Presented to the Society, January 24, 1969; received by the editors March 29, 1968.

⁽¹⁾ The author wishes to thank Professor Edward M. Brown for his help and encouragement. This work was partially supported by National Science Foundation Grant GP6526.

Proof. The universal covering space of a lens space is a 3-sphere (see, for example, 6.7.5 of Hilton and Wylie [4]). By (2.5) and (2.3), therefore, the lens spaces are in PC.

(2.7) Let M be a 2-manifold. Then the total space of any fibre bundle whose base space is the 1-sphere S^1 and whose fibre is M is in PC.

Proof. Such a total space is covered by $M \times R^1$, which is imbeddable in $M \times I$. By (2.4) and (2.2), $M \times R^1$ is in PC. By (2.3), therefore, the total space is in PC.

(2.8) The connected sum of two compact, orientable, connected 3-manifolds in PC is also in PC.

Proof. This follows from Theorem 1 of Milnor [5].

(2.9) If each of two compact, orientable connected 3-manifolds with connected nonvacuous boundary is in PC, then the 3-manifold obtained by pasting them together across a disk is also in PC.

Proof. This follows from the Decomposition Theorem of the author [2].

Example (2.9) is a special case of the present Theorem 1 of §4.

3. ***P*-admissibility.** The 3-manifolds considered in this paper are, in general, not connected. Thus, any result concerning the operation of pasting a 3-manifold to itself will also apply to the operation of pasting two disjoint 3-manifolds together.

CONVENTION. Let B be a 3-manifold which is obtained by pasting a 3-manifold M (connected or not) to itself across two disjoint homeomorphic 2-dimensional submanifolds of its boundary. It will always be assumed that the triangulation of B is sufficiently fine that the identification map $M \rightarrow B$ (which is induced by the pasting operation) is simplicial for some subdivision of the triangulation of M . In particular, the image in B of the pasting surfaces is assumed to be simplicially imbedded in B .

DEFINITION. Let R be a compact connected 2-manifold. One says that R is *P-admissible* if the following holds: For any 3-manifold B (connected or not) in PC and for any pair of imbeddings $f: R \rightarrow \text{bd}(B)$ and $g: R \rightarrow \text{bd}(B)$ whose images are disjoint, the 3-manifold obtained from B by identifying, for each point x in R , the points $f(x)$ and $g(x)$ of $\text{bd}(B)$ is in PC.

The main result of this paper may now be formulated as follows.

PASTING THEOREM. *The disk, the sphere, and the projective plane are P-admissible. Furthermore, if any other compact connected 2-manifold is P-admissible, then the Poincaré conjecture is true.*

4. **Disks are *P*-admissible.** The sole purpose of this section is the proof of Theorem 1. The numbered statements (4.1), ..., (4.6) are intermediate results in that proof.

THEOREM 1. *Let M be a 3-manifold (connected or not) in PC. Let D' and D'' be disjoint disks on $\text{bd}(M)$. And let B be a 3-manifold obtained from M by identifying D' and D'' under a homeomorphism. Then B is in PC.*

Proof. Let the disk D be the image in B of the disks D' and D'' . If a given homotopy 3-cell can be imbedded in B , then there is an imbedding of that homotopy 3-cell such that its image C lies in the interior of B and such that the components of $\text{bd}(C) \cap D$ are simple loops, each a crossing of surfaces. The theorem will be proved by an induction on the number of these loops.

Basis step. If $\text{bd}(C) \cap D$ is void, then $C \subset B - D = M - (D' \cup D'') \subset M$. Since M is in PC, the homotopy 3-cell C is a 3-cell.

Induction step. It will be shown that if $\text{bd}(C) \cap D$ has exactly n components, then C is a 3-cell.

The regular neighborhood theory of Whitehead [8] implies the existence of a regular neighborhood of $\text{bd}(C)$ and a regular neighborhood of D whose intersection is a regular neighborhood of $\text{bd}(C) \cap D$. The regular neighborhoods of $\text{bd}(C)$ and D will be realized here as the images of a pair of homeomorphisms $F: \text{bd}(C) \times [-1, 1] \rightarrow B$ and $H: D \times [-1, 1] \rightarrow B$ such that the following conditions hold (see Figure 1).

- (i) For all points x of $\text{bd}(C)$, $F(x, 0) = x$.
- (ii) $F(\text{bd}(C) \times \{-1\}) \subset C$.
- (iii) For all points y of D , $H(y, 0) = y$.
- (iv) $H(\text{bd}(D) \times [-1, 1]) = H(D \times [-1, 1]) \cap \text{bd}(B)$.
- (v) $F(\text{bd}(C) \times [-1, 1]) \cap H(D \times [-1, 1]) = H(F((\text{bd}(C) \cap D) \times [-1, 1]) \times [-1, 1])$.

In order to facilitate cutting and pasting the following additional condition is imposed.

- (vi) For all points x of $\text{bd}(C) \cap D$ and all points u, v of $[-1, 1]$, $F(H(x, u), v) = H(F(x, v), u)$.

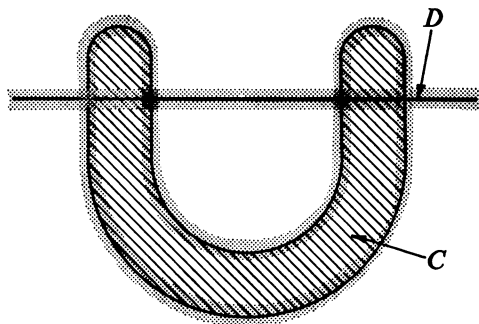


FIGURE 1

(4.1) If any component of $C \cap D$ is a disk, then C is a 3-cell.

Proof of (4.1). Let E be a component of $C \cap D$ which is a disk. The components of $C - H(E \times (-1, 1))$ are homotopy 3-cells, by the van Kampen theorem. There are fewer than n component loops in the intersection of the boundary of either component of $C - H(E \times (-1, 1))$ with the disk D . Therefore, by the induction

hypothesis, both components are 3-cells. Hence, C is a 3-cell (by Theorem 6 on p. III-19 of Zeeman [9], for example).

DEFINITION. Let M and N be manifolds, and let $f: M \rightarrow N$ be an imbedding. One says that f is a *proper imbedding* if $f(\text{bd}(M)) \subset \text{bd}(N)$ and $f(\text{int}(M)) \subset \text{int}(N)$.

The purpose of making the definitions of the next few paragraphs is to split the homotopy 3-cell C into two pieces, which will be called K_0 and K'_0 . The rest of the proof of the theorem consists of first showing that $K_0 \cap K'_0$ is a disk, properly imbedded in C (which implies, by the van Kampen theorem, that K_0 and K'_0 are homotopy 3-cells), and then showing that K_0 and K'_0 are 3-cells.

Because of statement (4.1), it will henceforth be assumed that no component of $C \cap D$ is a disk. Thus the components of $C \cap D$ are all disks-with-holes. Each such disk-with-holes is contained in a minimal subdisk of D . An *innermost* component of $C \cap D$ on the disk D is a component such that the minimal subdisk which contains it contains no other component of $C \cap D$.

Let D_0 be an innermost component of $C \cap D$ on the disk D . Let l_0, \dots, l_m be the components of $\text{bd}(D_0)$, with the loop l_0 outside of the loops l_1, \dots, l_m on D .

For $j=1, \dots, m$, let E_j be the disk on $\text{bd}(C)$ such that $\text{bd}(E_j)=l_j$ and $l_0 \notin E_j$. One assumes that the loops l_1, \dots, l_m are indexed so that if $j < i$, then the number of disks among E_1, \dots, E_m which contain E_j is not greater than the number of disks among E_1, \dots, E_m which contain E_i . Thus, the first few disks (i.e. those with lowest indices) are maximal as sets among E_1, \dots, E_m . Each of the next few disks is contained by exactly one member of the previous batch, and so on.

Since D_0 is a cycle in $C_2(C, \text{bd}(C); \mathbb{Z}_2)$, it must also be a boundary, because the homology of C is trivial. Therefore, $C - D_0$ has two components, whose closures will be denoted by K_m and K'_m , with K_m chosen so that it contains $H(D_0 \times [0, 1])$. For $j > 0$, given K_j , one defines $K_{j-1} = K_j \cup F(E_j \times [-j/n, 0])$ if $H(l_j \times [-1, 0]) \subset E_j$, or $K_{j-1} = K_j - F(E_j \times (-j/n, 0])$ if $H(l_j \times [0, 1]) \subset E_j$. Also, for $j > 0$, given K'_j , one defines $K'_{j-1} = K'_j - F(E_j \times (-j/n, 0])$ if $H(l_j \times [-1, 0]) \subset E_j$, or $K_{j-1} = K'_j \cup F(E_j \times [-j/n, 0])$ if $H(l_j \times [0, 1]) \subset E_j$ (see Figure 2).

For $j=0, \dots, m-1$, define R_j to be the union of the disk-with-holes

$$D_0 - (F(l_{j+1} \times (-(j+1)/n, 0])) \cup \dots \cup F(l_m \times (-m/n, 0]))$$

with the disks $F(E_{j+1} \times \{-(j+1)/n\}), \dots, F(E_m \times \{-m/n\})$. Also, define $R_m = D_0$.

(4.2) For $j=0, \dots, m$, R_j is a disk-with- j -holes.

Proof of (4.2). First of all, R_m is a disk-with- m -holes. because $R_m = D_0$. Now for a given positive index $j \leq m$, suppose that R_j is a disk-with- j -holes. Since $\text{bd}(C) \cap D_0 = l_0 \cup \dots \cup l_m$ and $E_j \subset \text{bd}(C) - l_0$, it follows that $E_j \cap D_0 \subset l_1 \cup \dots \cup l_m$. If $l_i \subset E_j$, then $E_i \subset E_j$, by definition of the disks E_i . Thus by the indexing criterion for the disks E_i , $E_j \cap D_0 \subset l_j \cup \dots \cup l_m$. It follows that the intersection of the disk $F(E_j \times \{-j/n\})$ with $D_0 - (F(l_j \times (-j/n, 0])) \cup \dots \cup F(l_m \times (-m/n, 0]))$ is the loop $F(l_j \times \{-j/n\})$. Evidently, the intersection of $F(E_j \times \{-j/n\})$ with each of the disks $F(E_{j+1} \times \{-(j+1)/n\}), \dots, F(E_m \times \{-m/n\})$ is void. Therefore, the intersection of

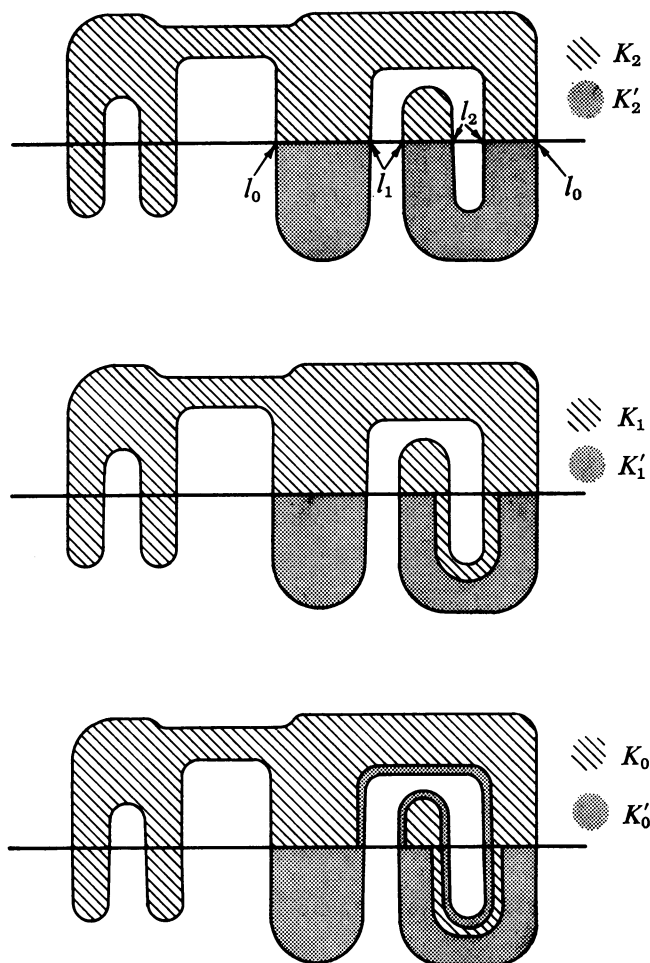


FIGURE 2

the disk $F(E_j \times \{-j/n\})$ with $R_j - F(l_j \times (-j/n, 0])$ is the loop $F(l_j \times \{-j/n\})$. Since R_j is a disk-with- j -holes, so is $R_j - F(l_j \times (-j/n, 0])$. Since R_{j-1} is the union of $F(E_j \times \{-j/n\})$ and $R_j - F(l_j \times (-j/n, 0])$, it must be a disk-with- $(j-1)$ -holes.

(4.3) For $j=0, \dots, m$, $K_j \cap K'_j = R_j$.

Proof of (4.3). Since K_m and K'_m are the closures of the two components of $C - R_m$, it follows that $K_m \cap K'_m = R_m$. Now for a given positive index $j \leq m$, suppose that $K_j \cap K'_j = R_j$. There are two cases, depending upon whether $H(l_j \times [-1, 0]) \subset E_j$ or $H(l_j \times [0, 1]) \subset E_j$. The symmetry of the definitions for K_j and K'_j assures one that it is sufficient to consider only one of these cases. So suppose that $H(l_j \times [-1, 0]) \subset E_j$. From the indexing condition on the disks E_1, \dots, E_m it follows that some collection E_{j+1}, \dots, E_r contains all the disks among E_{j+1}, \dots, E_m which are maximal as sets among E_{j+1}, \dots, E_m . One permutes the indices $j+1, \dots, r$ so that for $i=j+1, \dots, r$, $E_i \subset E_j$ if and only if $i \leq s$.

One observes that $\text{cl}(E_j - (E_{j+1} \cup \dots \cup E_s))$ is a disk-with-holes whose boundary components are the loops l_j, \dots, l_s and whose intersection with $R_j (= K_j \cap K'_j)$ is the union of the loops l_j, \dots, l_s . Since $H(l_j \times [-1, 0]) \subset E_j$, it follows from the construction of K'_j that $\text{cl}(E_j - (E_{j+1} \cup \dots \cup E_s))$ lies in K'_j and that $H(l_j \times [0, 1]) \subset K_j$. It will now be shown that $H(l_{j+1} \times [0, 1]) \subset E_{j+1}$. Otherwise, $F(E_{j+1} \times [-j/n, 0]) \subset K_j$, from the construction of K_j . Let b be a path in the interior of E_j from a point in $E_j - (E_{j+1} \cup \dots \cup E_s)$ to a point in the interior of E_{j+1} . By the construction of R_j , the path b never crosses R_j . But it goes from a point in K'_j to a point in K_j . This is a contradiction. Similarly, for $i=j+2, \dots, s$, $H(l_i \times [0, 1]) \subset E_i$.

It now follows that for $i=j+1, \dots, s$, the 3-cell $F(E_i \times [-i/n, 0])$ lies in K'_j . Therefore, $F(E_j \times [-(j+\frac{1}{2})/n, 0]) \subset K'_j$, which implies that $F(E_j \times [-j/n, 0]) \cap K_j = F(l_j \times [-j/n, 0])$. Hence

$$\begin{aligned} K_{j-1} \cap K'_{j-1} &= (K_j \cup F(E_j \times [-j/n, 0])) \cap (K'_j - F(E_j \times (-j/n, 0])) \\ &= (K_j \cap (K'_j - F(E_j \times (-j/n, 0)))) \cup (F(E_j \times [-j/n, 0]) \\ &\quad \cap (K'_j - F(E_j \times (-j/n, 0)))) \\ &= (R_j - F(l_j \times (-j/n, 0])) \cup F(E_j \times \{-j/n\}) = R_{j-1}. \end{aligned}$$

Thus, statement (4.3) is proved.

(4.4) K_0 and K'_0 are homotopy 3-cells whose union is the homotopy 3-cell C and whose intersection is the disk R_0 , which is properly imbedded in C .

Proof of (4.4). By their constructions, the union of K_0 and K'_0 is C . By statement (4.2), R_0 is a disk. By its construction, R_0 is properly imbedded in C . By statement (4.3), $K_0 \cap K'_0 = R_0$. By the van Kampen theorem, K_0 and K'_0 must be homotopy 3-cells.

It will soon be proved that K_0 and K'_0 are 3-cells. For this, it will be convenient to define certain new objects.

For $j=1, \dots, m$, let D_j be the subdisk of D such that $\text{bd}(D_j) = l_j$.

Let $L_m = K_m$ and let $L'_m = K'_m$. For $j > 0$, given L_j , one defines $L_{j-1} = L_j \cup H(D_j \times [0, \frac{1}{2}])$ if $H(l_j \times [-1, 0]) \subset E_j$ and $L_{j-1} = L_j$ if $H(l_j \times [0, 1]) \subset E_j$. Also, for $j > 0$, given L'_j , one defines $L'_{j-1} = L'_j$ if $H(l_j \times [-1, 0]) \subset E_j$ and $L'_{j-1} = L'_j \cup H(D_j \times [-\frac{1}{2}, 0])$ if $H(l_j \times [0, 1]) \subset E_j$ (see Figure 3).

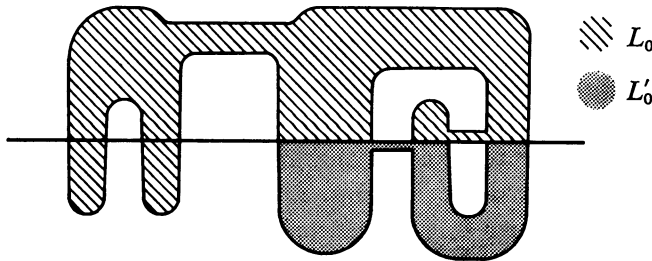


FIGURE 3

(4.5) For $j=0, \dots, m$, there is a homeomorphism $f_j: K_j \rightarrow L_j$. In particular, K_0 and L_0 are homeomorphic homotopy 3-cells.

Proof of (4.5). This proof will be accomplished by a lot of cutting and pasting operations, and the amount of detail is regrettably large. An examination of K_0 , K_1 , and K_2 in Figure 2 reveals that they are homeomorphic to L_0 , L_1 , and L_2 , respectively, in Figure 3. In what follows the correspondence visible in the figures will be described rigorously.

For $1 \leq i \leq m$, for $y \in I_i$, for $-1 \leq u \leq 0$, and for $0 \leq v \leq 1$, let $h_i(F(H(y, v), u)) = (y, v, u)$ define a homeomorphism from $F(H(I_i \times [0, 1]) \times [-1, 0])$ onto $I_i \times [0, 1] \times [-1, 0]$, and let g_i be the imbedding of $(I_i \times [0, 1] \times \{-1\}) \cup (I_i \times \{0, 1\} \times [-1, 0])$ into $\text{bd } (F(H(I_i \times [0, 1]) \times [-1, 0]))$ which is given as follows: let $g_i(y, v, -1) = F(H(y, v), -1)$, let $g_i(u, 1, u) = F(H(y, 1), u)$, if $-1 \leq u \leq -i/n$ then let $g_i(y, 0, u) = F(H(y, 0), (nu+i)/(n-i))$, and if $-i/n \leq u \leq 0$ then let $g_i(y, 0, u) = F(H(y, (i+nu)/2i), 0)$. By the Product Theorem of Brown [1], each map g_i extends to a homeomorphism of $I_i \times [0, 1] \times [-1, 0]$ onto $F(H(I_i \times [0, 1]) \times [-1, 0])$, which will be denoted by g'_i . It is clear that the autohomeomorphism $g'_i h_i$ on $F(H(I_i \times [0, 1]) \times [-1, 0])$ takes $F(I_i \times [-i/n, 0])$ onto $H(I_i \times [0, \frac{1}{2}])$ and leaves $F(H(I_i \times [0, 1]) \times [-1]) \cup F(H(I_i \times \{1\}) \times [-1, 0])$ fixed.

Define $f_m: K_m \rightarrow L_m$ to be the homeomorphism which is the identity on the complement in K_m of the union of $F(H(I_1 \times [0, 1]) \times [-1, 0]), \dots, F(H(I_m \times [0, 1]) \times [-1, 0])$ and which is the map $g'_i h_i$ on each of the solid tori $F(H(I_i \times [0, 1]) \times [-1, 0])$. For a positive index $j \leq m$ suppose that a homeomorphism $f_j: K_j \rightarrow L_j$ has been defined so that if $1 \leq i \leq j$ then $f_j(F(I_i \times [-i/n, 0])) = H(I_i \times [0, \frac{1}{2}])$.

Suppose that $H(I_j \times [-1, 0]) \subset E_j$. Then $K_{j-1} = K_j \cup F(E_j \times [-j/n, 0])$ and $L_{j-1} = L_j \cup (D_j \times [0, \frac{1}{2}])$. It was shown in the proof of statement (4.3) that $K_j \cap F(E_j \times [-j/n, 0]) = F(I_j \times [-j/n, 0])$. It is evident that $L_j \cap H(D_j \times [0, \frac{1}{2}]) = H(I_j \times [0, \frac{1}{2}])$. The identity map $I_j \rightarrow I_j$ extends to a homeomorphism $E_j \rightarrow D_j$ which extends to homeomorphism $p_j: F(E_j \times [-j/n, 0]) \rightarrow H(D_j \times [0, \frac{1}{2}])$ such that $p_j(F(I_j \times [-j/n, 0])) = H(I_j \times [0, \frac{1}{2}])$. The map $f_{j-1}: K_{j-1} \rightarrow L_{j-1}$ is defined to be f_j on K_j and p_j on $F(E_j \times [-j/n, 0])$. Thus, f_{j-1} is a homeomorphism from K_{j-1} onto L_{j-1} such that if $1 \leq i \leq j-1$ then $f_{j-1}(F(I_i \times [-i/n, 0])) = H(I_i \times [0, \frac{1}{2}])$.

Now suppose, on the other hand, that $H(I_j \times [0, 1]) \subset E_j$. Then $K_{j-1} = K_j - F(E_j \times (-j/n, 0])$ and $L_{j-1} = L_j$. It is evident from the proof of statement (4.3) that $F(E_j \times [-(j+\frac{1}{2})/n, 0]) \subset K_j$, which implies that $K_{j-1} \cap F(E_j \times [-j/n, 0])$ is the disk $F(E_j \times \{-j/n\})$. The set $F(I_1 \times [-1/n, 0]) \cup \dots \cup F(I_{j-1} \times [-(j-1)/n, 0])$ is closed and it is disjoint from $F(E_j \times [-j/n, 0])$ because of the indexing criterion on the disks E_i . Thus, there is a homeomorphism $q_j: K_{j-1} \rightarrow K_j$ which is the identity on that set (see Theorem 6 on p. III-19 of Zeeman [9], for example). Define $f_{j-1}: K_{j-1} \rightarrow L_{j-1}$ to be the homeomorphism $f_j q_j$. The definition of q_j assures one that if $1 \leq i \leq j-1$ then $f_{j-1}(F(I_i \times [-i/n, 0])) = H(I_i \times [0, \frac{1}{2}])$. Thus, statement (4.5) is proved.

Correspondingly, one also proves statement (4.5)'.

(4.5)' For $j=0, \dots, m$, there is a homeomorphism $f'_j: K'_j \rightarrow L'_j$. In particular, K'_0 and L'_0 are homeomorphic homotopy 3-cells.

(4.6) The homotopy 3-cells L_0 and L'_0 are 3-cells.

Proof of (4.6). By the symmetry of the definitions of L_0 and L'_0 , it suffices to show that L_0 is a 3-cell. To this intent, let D^* denote the disk $D_0 \cup D_1 \cup \dots \cup D_m$. One observes that D^* is the smallest subdisk of D which contains the disk-with-holes D_0 . Also, let $L_0^* = L_0 - H(D^* \times [0, \frac{1}{4}))$, so L_0^* is homeomorphic to L_0 . The boundary of L_0^* is in general position with respect to the disk D and it has at least $m+1$ fewer components of intersection (representing the loops l_0, \dots, l_m) with D than $\text{bd}(C)$ has. By the induction hypothesis, L_0^* is a 3-cell. Hence, so is L_0 .

Since $K_0 \approx L_0$ and $K'_0 \approx L'_0$ and $K_0 \cap K'_0$ is the disk R_0 , which is properly imbedded in C , and since $C = K_0 \cup K'_0$, the homotopy 3-cell C is a 3-cell. Hence, the 3-manifold B is in PC.

5. Spheres and projective planes are P -admissible.

THEOREM 2. *Let M be a 3-manifold (connected or not) in PC. Let S' and S'' be two distinct components of $\text{bd}(M)$ which are both spheres. And let B be a 3-manifold obtained from M by identifying S' and S'' under homeomorphism. Then B is in PC.*

Proof. Let D' and D'' be disks on S' and S'' respectively such that the given homeomorphism $S' \rightarrow S''$ carries D' onto D'' . Let B' be the 3-manifold obtained from M by identifying D' and D'' under the restriction of the homeomorphism $S' \rightarrow S''$. By Theorem 1, B' is in PC. The 3-manifold B'' obtained from B' by filling in a 3-cell along the boundary component $\text{cl}(S' - D') \cup \text{cl}(S'' - D'')$ is homeomorphic to B . By Theorem 3 ("homogeneity") of Eugenheim [3], a homotopy 3-cell can be imbedded in B'' only if it can be imbedded in B' . Hence, B'' is in PC and, consequently, so is B .

THEOREM 3. *Let M be a 3-manifold (connected or not) in PC. Let P' and P'' be two distinct components of $\text{bd}(M)$ which are both projective planes. And let B be a 3-manifold obtained from M by identifying P' and P'' under a homeomorphism. Then B is in PC.*

Proof. Let the projective plane P be the image in B of the projective planes P' and P'' . If a given homotopy 3-cell can be imbedded in B , then there is an imbedding of that homotopy 3-cell such that its image C lies in the interior of B and such that the components of $\text{bd}(C) \cap P$ are simple loops, each a crossing of surfaces.

The first thing to be established is that each component of $\text{bd}(C) \cap P$ is a disk or a disk-with-holes. The alternative, of course, is that some component R of $\text{bd}(C) \cap P$ is a nonorientable 2-manifold. One observes that R is properly imbedded in the homotopy 3-cell C , and that R is a cycle in $C_2(C, \text{bd}(C); \mathbb{Z}_2)$. Since the homology of C is trivial, R must be a boundary, which implies that C contains a nonorientable 3-dimensional submanifold, which is a contradiction.

Thus, it is established that each component of $\text{bd}(C) \cap P$ is a disk or a disk-with-holes.

The next fact needed is that there is a disk on the projective plane P which contains $C \cap P$ in its interior. Let R_1, \dots, R_s be the components of $C \cap P$. For $i=1, \dots, s$, it follows from the classification of compact 2-manifolds that exactly one component of $P - \text{int}(R_i)$ is a Möbius band and every other component is a disk. For $j=1, \dots, s$, let D_j be the disk which is the union of R_j and all the components of $P - \text{int}(R_j)$ which are disks. Let D be a disk on P which contains $D_1 \cup \dots \cup D_s$ in its interior. Then the disk D contains $C \cap P$ in its interior.

Now let D' and D'' be the pre-images of the disk D in the projective planes P' and P'' respectively. The homotopy 3-cell C can evidently be imbedded in the 3-manifold B' obtained from M by identifying D' and D'' under the restriction of the homeomorphism $P' \rightarrow P''$ which is given in the hypothesis. By Theorem 1, the 3-manifold B' is in PC. Hence, C is a 3-cell. Hence, the 3-manifold B is in PC.

6. P -admissibility and the Poincaré conjecture. The first statement of the Pasting Theorem of §3 has been established via Theorems 1, 2, and 3. The second statement of the Pasting Theorem will be proved in this section as Theorems 4 and 5.

Let B be a 3-manifold in PC and let R be a compact connected 2-dimensional submanifold of $\text{bd}(B)$. One says that (B, R) is a *P -admissible pair* if the following holds: For any 3-manifold B' (disjoint from B) in PC and for any imbedding $f: R \rightarrow \text{bd}(B')$, the 3-manifold obtained from $B \cup B'$ by identifying, for each point x in R , the points x and $f(x)$ in $\text{bd}(B \cup B')$ is in PC.

A *solid torus* is a 3-manifold which is homeomorphic to the product of a disk and a 1-sphere.

LEMMA 1. *Let T be a solid torus. If the pair $(T, \text{bd}(T))$ is P -admissible, then the Poincaré conjecture is true.*

Proof. Let N be any compact, orientable, connected 3-manifold with vacuous boundary. By Theorem 6 of Wallace [7], a homeomorphic copy of N can be constructed by removing from the 3-sphere S^3 a collection of disjoint solid tori and sewing them back differently. The closure of the complement in S^3 of the union of the solid tori is in PC because it is a submanifold of S^3 . If the pair $(T, \text{bd}(T))$ is P -admissible, then any 3-manifold which results from sewing the tori back is in PC. Thus, every compact, orientable, connected 3-manifold with vacuous boundary would be in PC. This would imply, of course, that the Poincaré conjecture is true.

LEMMA 2. *Let K be a 3-cell and let A be an annulus (i.e. $A \approx S^1 \times [0, 1]$) on $\text{bd}(K)$. If the pair (K, A) is P -admissible, then the Poincaré conjecture is true.*

Proof. Let B be a 3-manifold in PC which has a boundary component R such that R is a torus. Let T be a solid torus disjoint from B and $h: \text{bd}(T) \rightarrow R$ a

homeomorphism. It will be shown that if (K, A) is P -admissible, then the 3-manifold B_1 obtained from $B \cup T$ by identifying $\text{bd}(T)$ and R under h is in PC, which implies that $(T, \text{bd}(T))$ is a P -admissible pair, which implies by Lemma 1 that the Poincaré conjecture is true.

Suppose that (K, A) is P -admissible. Let $g: K \rightarrow T$ be an imbedding such that $g(K) \cap \text{bd}(T) = g(A)$ and $\text{cl}(T - g(K))$ is a 3-cell. Since (K, A) is P -admissible, the 3-manifold B_2 obtained from $B \cup g(K)$ by identifying $g(A)$ and $hg(A)$ via h is in PC. The 3-manifold B_3 obtained from B_2 by filling in a 3-cell along the 2-sphere $(R - hg(A)) \cup hg(\text{bd}(K) - \text{int}(A))$ is in PC, by Theorem 3 of Gugenheim [3]. Since B_1 and B_3 are homeomorphic, B_1 is in PC, which implies that the Poincaré conjecture is true.

THEOREM 4. *If any orientable 2-manifold which is neither a disk nor a sphere is P -admissible, then the Poincaré conjecture is true.*

Proof. Statements (6.1) and (6.2) are intermediate results.

(6.1) Let V and W be orientable compact connected 2-manifolds whose boundaries have the same number of components and such that $\text{genus}(W) = \text{genus}(V) + 1$. Let W be P -admissible. Then V is P -admissible.

Proof of (6.1). Let M be a 3-manifold in PC. Let V' and V'' be two disjoint homeomorphic copies of V on $\text{bd}(M)$. And let $h: V' \rightarrow V''$ be a homeomorphism. It will be shown that the 3-manifold B obtained by identifying V' and V'' under h is in PC.

Let T be a solid torus in M whose intersection with $\text{bd}(M)$ is a disk in the interior of V' (see Figure 4a). Let U be a solid torus which is disjoint from M , and let $g: T \rightarrow U$ be a homeomorphism. Let M_1 be the 3-manifold obtained from $\text{cl}(M - T)$ by pasting the solid torus U to $\text{cl}(M - T)$ at the disk $h(T \cap \text{bd}(M))$ via the composition of h restricted to $T \cap \text{bd}(M)$ and g^{-1} restricted to the disk $g(T \cap \text{bd}(M))$ on $\text{bd}(U)$ (see Figure 4b).

Since $\text{cl}(M - T) \subset M$, $\text{cl}(M - T)$ is in PC. By Theorem 1, the 3-manifold M_1 is in PC. Let $W' = (V' \cup \text{bd}(T)) - \text{int}(T \cap V')$ and $W'' = (V'' \cup \text{bd}(U))$

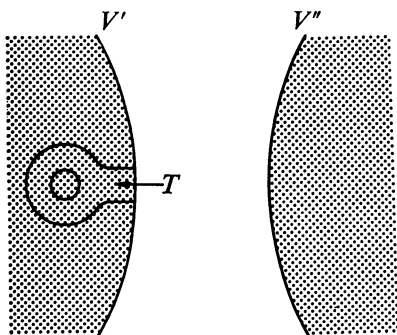


FIGURE 4a

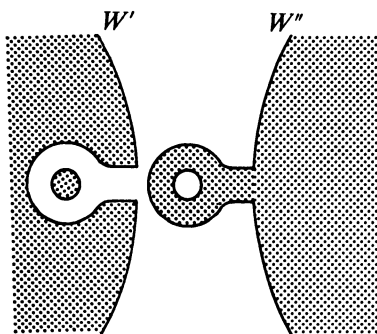


FIGURE 4b

$-\text{int}(h(T \cap V'))$). So W' and W'' are homeomorphic to the P -admissible 2-manifold W . Thus, the 3-manifold M_2 which is obtained from M_1 by identifying $W' - \text{bd}(T) = V' - \text{bd}(T)$ with $W'' - \text{bd}(U) = V'' - h(\text{bd}(T))$ via h and by identifying $W' \cap \text{bd}(T) = \text{bd}(T) - V'$ with $W'' \cap \text{bd}(U) = h(\text{bd}(T) - V'')$ under g is in PC. Evidently, B is homeomorphic to M_2 . Hence, the 3-manifold B is in PC, which proves that the 2-manifold V is P -admissible.

(6.2) Let V and W be orientable compact connected 2-manifolds which have the same genus and such that $\text{bd}(W)$ has one more component than $\text{bd}(V)$. Let W be P -admissible. Then V is P -admissible.

The proof of statement (6.2) is omitted, since it is an easy generalization of the proof of Theorem 2. The Poincaré conjecture is true if the torus is P -admissible, by Lemma 1. By statement (6.1), it is true if any compact orientable 2-manifold with positive genus and vacuous boundary is P -admissible. In fact, by statement (6.2), it is also true if any compact orientable 2-manifold with positive genus and nonvacuous boundary is P -admissible. The Poincaré conjecture is true if the annulus is P -admissible, by Lemma 2. By statement (6.2), it is true if any disk-with-holes is P -admissible. This completes the proof of Theorem 4.

LEMMA 3. Let R be a Möbius band. Then the pair $(R \times [0, 1], \text{bd}(R) \times [0, 1])$ is P -admissible.

Proof. Let M be a 3-manifold in PC and let $h: \text{bd}(R) \times [0, 1] \rightarrow \text{bd}(M)$ be an imbedding. It will be shown that the 3-manifold B obtained from $M \cup (R \times [0, 1])$ by identifying the domain and the image of h is in PC. Let k be an orientation reversing loop on R and let A be the annulus $k \times [0, 1]$ in $R \times [0, 1]$. If a given homotopy 3-cell can be imbedded in B , then there is an imbedding of that homotopy 3-cell such that its image C lies in the interior of B and such that the components of $\text{bd}(C) \cap A$ are simple loops, each a crossing of surfaces.

Each component S of $\text{bd}(C) \cap A$ is orientable, for homological reasons given in the proof of Theorem 3, so it must be a disk or a disk-with-holes. None of the boundary components of S can separate $k \times 0$ from $k \times 1$ because then $k \times 0$ and $k \times 1$ would be freely homotopic to that boundary component of S . But each component of $\text{bd}(S)$ is trivial, because S lies in the homotopy 3-cell C . This would imply that $k \times 0$ and $k \times 1$ are trivial in B , which is a contradiction. It follows that S lies on a disk which is a submanifold of the annulus A .

Let T be a tubular neighborhood of the loop k in the interior of the Möbius band R , chosen so that the components of $\text{bd}(C) \cap (\text{bd}(T) \times [0, 1])$ are simple loops, each a crossing of surfaces. One observes that T is a Möbius band and that $\text{cl}(B - (T \times [0, 1]))$ is homeomorphic to the 3-manifold M and, therefore, in PC. Since each component of $C \cap (\text{bd}(T) \times [0, 1])$ lies on a disk on the annulus $\text{bd}(T) \times [0, 1]$, one may prove that C is a 3-cell by applying Theorem 1. That is, take D to be a disk in the interior of $\text{bd}(T) \times [0, 1]$ such that $C \cap (\text{bd}(T) \times [0, 1])$ lies in the interior of D . Then C lies in the 3-manifold B' obtained from

$\text{cl}(B - (T \times [0, 1]))$ and $T \times [0, 1]$ by cutting B across $(\text{bd}(T) \times [0, 1]) - D$. By Theorem 1, B' is in PC.

THEOREM 5. *If any nonorientable 2-manifold which is not a projective plane is P -admissible, then the Poincaré conjecture is true.*

Proof. The following statement is the main fact needed.

(6.3) Let V and W be compact connected 2-manifolds such that one can obtain a homeomorphic copy of V by removing the interior of a Möbius band from W . Let W be P -admissible. Then V is P -admissible.

Proof of (6.3). Let M be a 3-manifold in PC. Let V' and V'' be two disjoint homeomorphic copies of V on $\text{bd}(M)$. And let $h: V' \rightarrow V''$ be a homeomorphism. It will be shown that the 3-manifold B obtained by identifying V' and V'' under h is in PC.

Let k be a component of $\text{bd}(V')$ and let A be a regular neighborhood of the loop k in $\text{cl}(\text{bd}(M) - V') - V''$ such that $h(A) \cap V'$ is empty. Let R be a Möbius band. And let M_1 be the 3-manifold obtained from $M \cup (R \times [0, 1]) \cup (R \times [2, 3])$ by identifying the annulus A and $\text{bd}(R) \times [0, 1]$ under a homeomorphism which takes $\text{bd}(R) \times \{0\}$ onto the loop k and by identifying the annulus $h(A)$ and $\text{bd}(R) \times [2, 3]$ under a homeomorphism which takes $\text{bd}(R) \times \{2\}$ onto the loop $h(k)$ (see Figure 5).

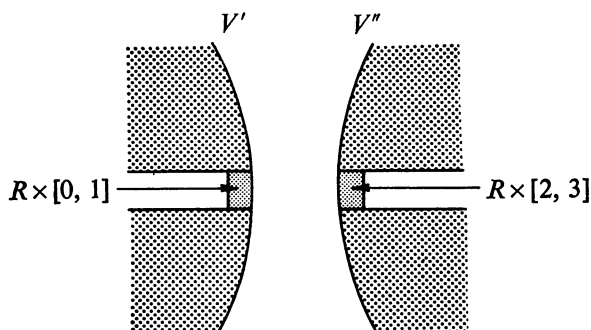


FIGURE 5

The 3-manifold M_1 is in PC, because of Lemma 3. Let $W' = V' \cup (R \times \{0\})$ and let $W'' = V'' \cup (R \times \{2\})$. Then W' and W'' are homeomorphic to the 2-manifold W . Let $f: W' \rightarrow W''$ be a homeomorphism which agrees with $h: V' \rightarrow V''$ on V' . Then let B_1 be the 3-manifold obtained from M_1 by identifying W' and W'' under f . Since W is P -admissible, B_1 is in PC. Since $B \subset B_1$, it follows that B is in PC. Hence, B is P -admissible.

The proof of Theorem 5 is now easily completed. Let Q be any nonorientable, compact, connected 2-manifold. If m is the number of crosscaps on Q and r is the number of components of $\text{bd}(Q)$, it follows from m applications of statement (6.3)

that Q is P -admissible only if the disk-with- $(m+r-1)$ -holes is P -admissible. By Theorem 4, therefore, if any nonorientable compact connected surface other than a projective plane is P -admissible, the Poincaré conjecture is true.

7. Added in proof. Let M be a 3-manifold, let R be a compact, connected 2-dimensional submanifold of $\text{bd}(M)$, and suppose that the component M' of M which contains R is not a 3-cell. One says that R is *incompressible* if the kernel of the induced map $\pi_1(R) \rightarrow \pi_1(M')$ is trivial. C. D. Feustel and the author have jointly observed that the proof of Theorem 1 can be easily modified to give a proof of the following theorem.

THEOREM 6. *Let M be a 3-manifold (connected or not) in PC. Let R' and R'' be disjoint, homeomorphic, incompressible surfaces in $\text{bd}(M)$, and let B be a 3-manifold obtained from M by identifying R' and R'' under a homeomorphism. Then B is in PC.*

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